Topological Dimension of Shapes from Discrete Samples

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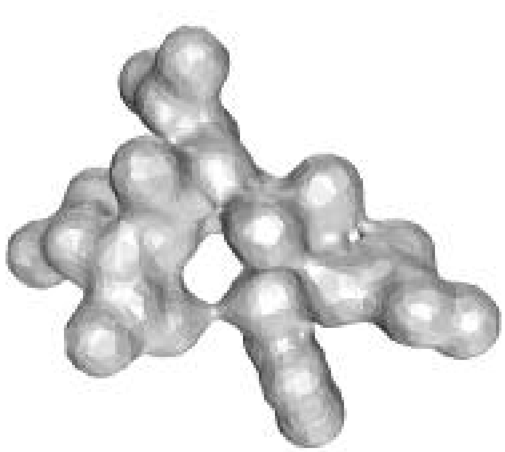
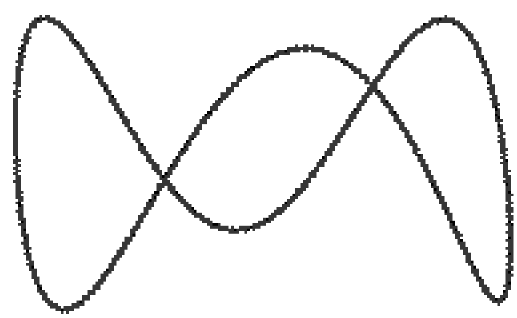
Shapes

• Shapes for us are smooth compact manifolds embedded in an Euclidean space.

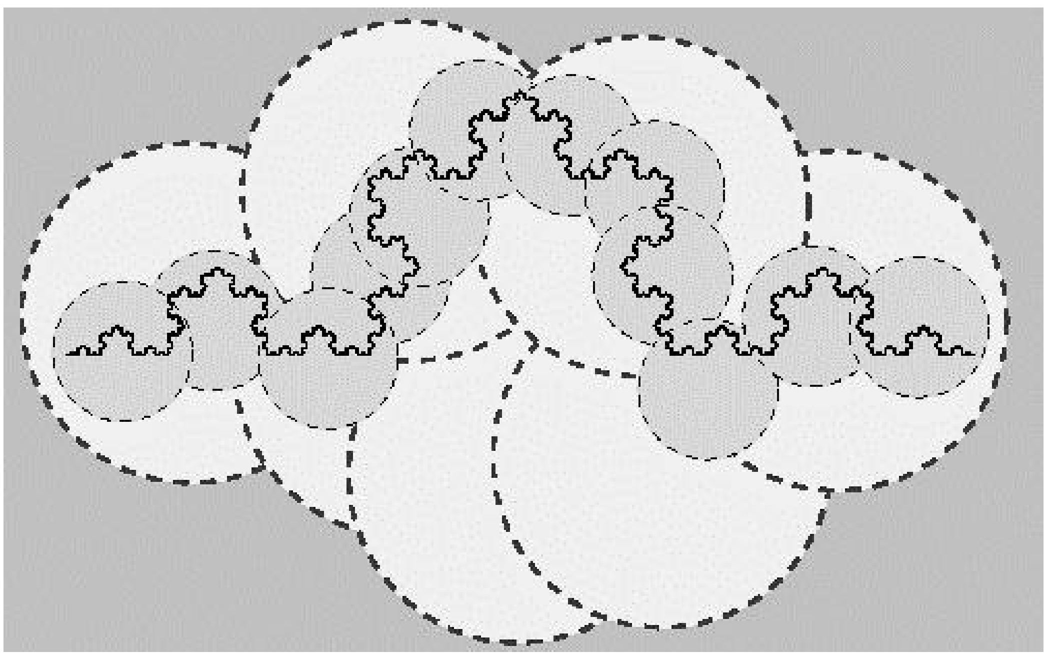




Topological Dimension

•C′ is a refinement of covering C of X

• X has topological dimension d if each point in X is covered with at most d+1 sets in C′ and d is the smallest

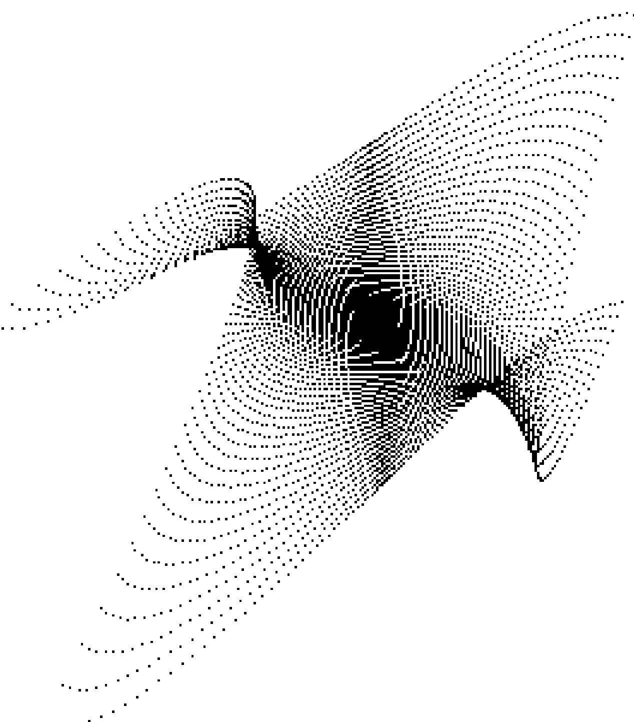


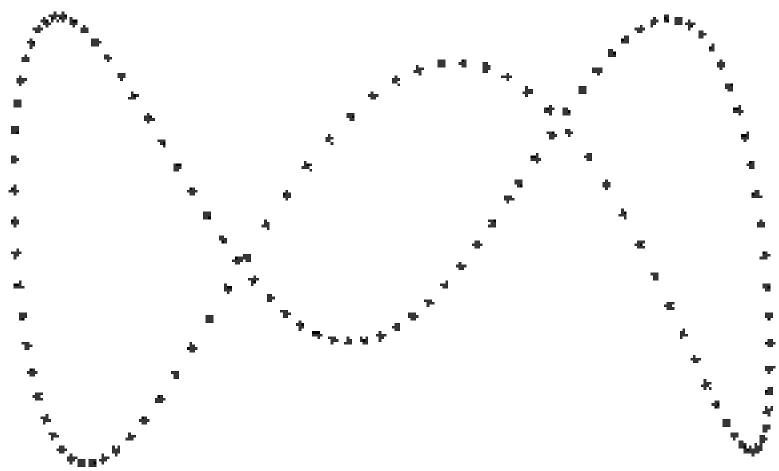
Koch curve:

Topological dimension

Fractal dimension: 1.26

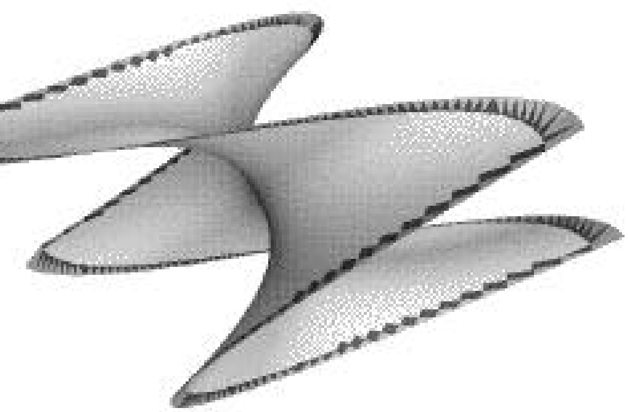
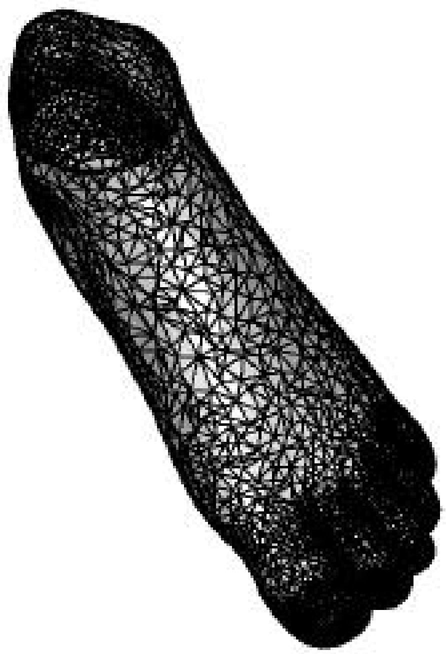
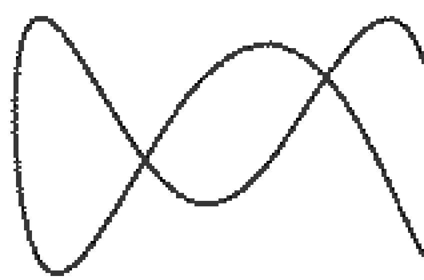
Sampling



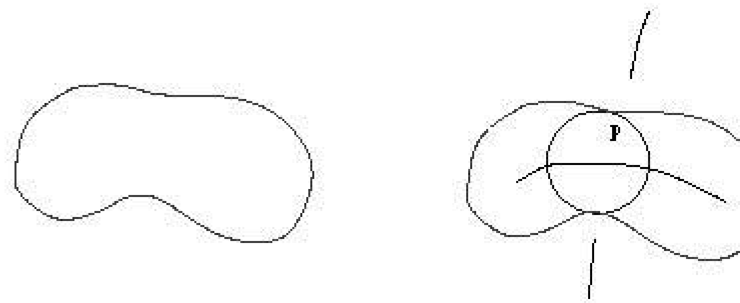


Motivations

- Manifold learning
- Shape reconstruction

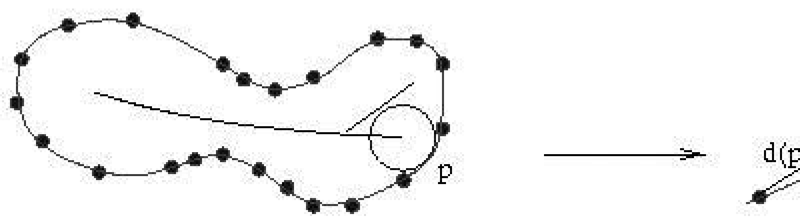




Local feature size and sampling



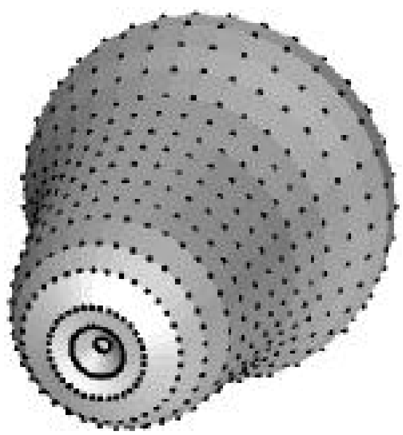
- Medial axis A
- Local feature size f : $R^d \rightarrow R$,

f(p) is the smallest distance to A



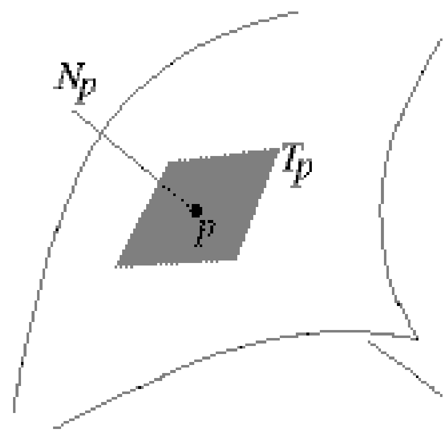
- ε-sampling [ABE98]
- d(p): Euclidean distance of p to nearest sample
- $\varepsilon \geq d(p)/f(p)$

Sampling and Ambiguity

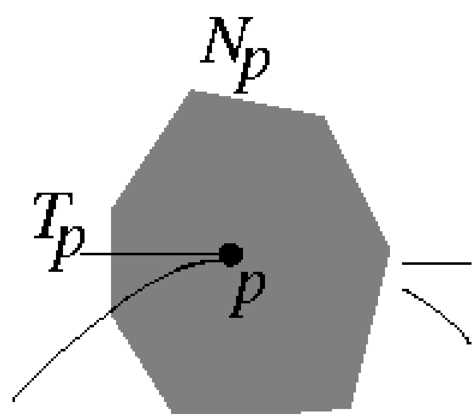


- (ε,δ)-sampling
- ε-sampling and all samples are > δf(p) away
- ε/2 < δ < ε

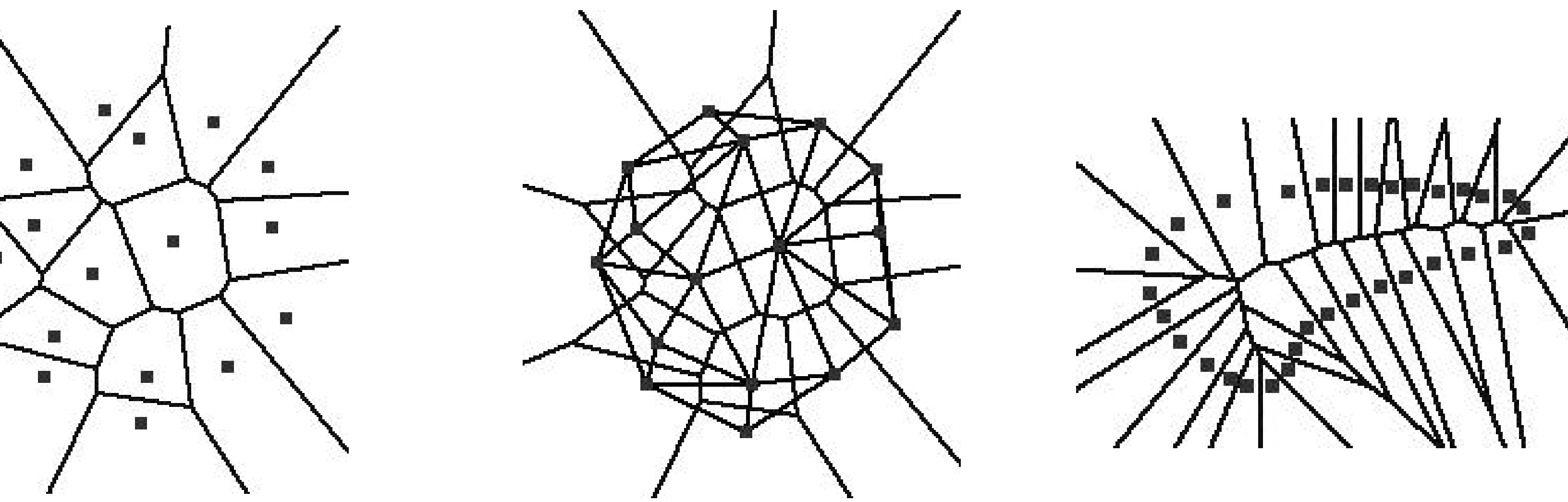
Tangent and Normal Spaces



- Space spanned by tangents T(p)
- Space spanned by normals N(p)



Voronoi Diagram / Delaunay triangulation



Tangent and Normal Polytopes

- $T^{\circ}(p) = V(p) \cap T(p)$
- $N^{\circ}(p) = V(p) \cap N(p)$

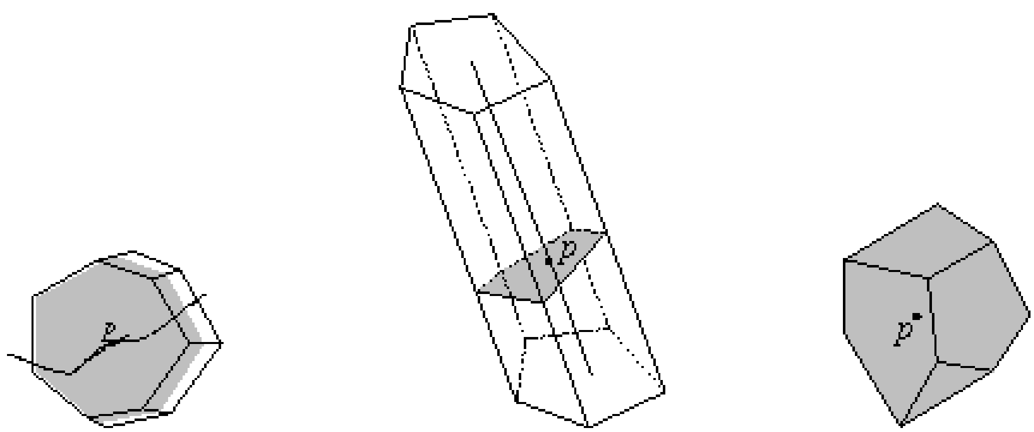
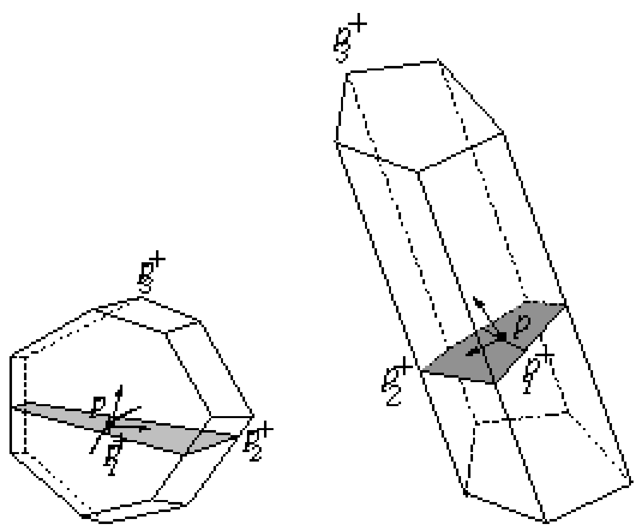


Figure 2: Tangent and normal polytopes of a sample on a curve (left), surface (middle) and solid (right).

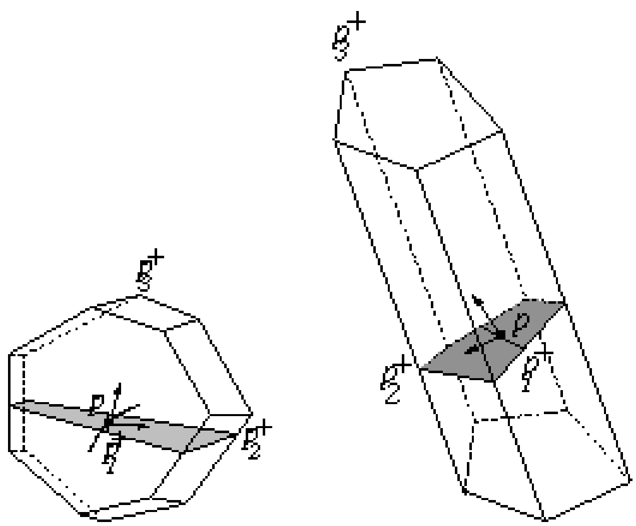
Voronoi Subpolytopes

- $V(p,i) \quad V(p),\ 1 \leq i \leq d$
- pole p_i^+
- pole vector $v(p,i)$



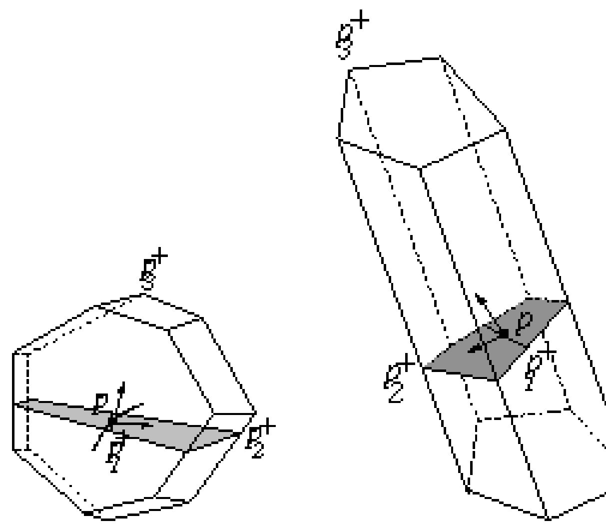
Heights

- pole vector v(p,i)
- heights H(p,i)= ||v(p,i)||



Idea

- M is a k-manifold in R^d, P is an (ε,δ)-sample
- heights H(p,i) are small for $1 \leq i \leq k$ and big for $k > i$
- Compare with H(p,1)



Normal Lemma

Lemma 1 *If* $v \in V(p,i)$ *for* $k < i \leq d$ *and* $||v - p|| > \mu f(p)$. *Then* $\exists \mathbf{n}(p) \in N(p)$ *so that*

$$\angle(\mathbf{n}(p),(v-p)) \leq \sin^{-1}\frac{\varepsilon}{\mu(1-\varepsilon)} + \sin^{-1}\frac{\varepsilon}{1-\varepsilon}.$$

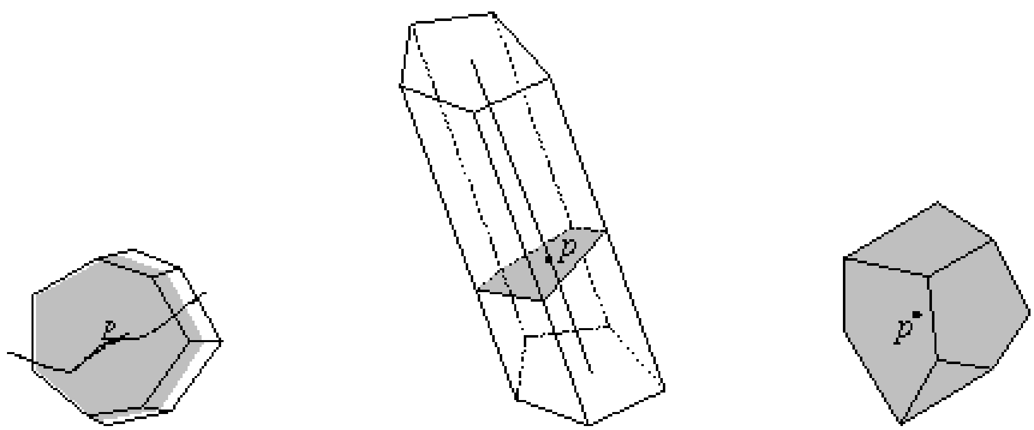
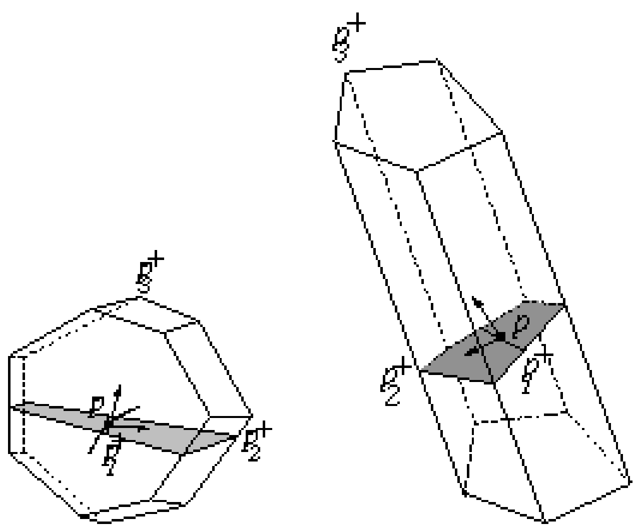


Figure 2: Tangent and normal polytopes of a sample on a curve (left), surface (middle) and solid (right).

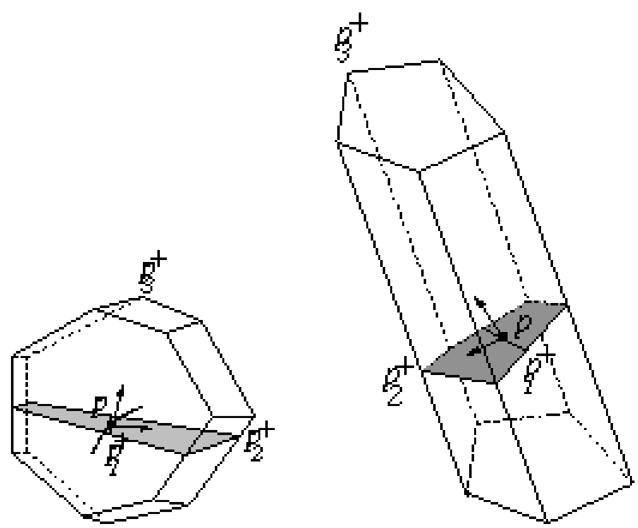
Height Lemma

Lemma 2 $H(p, i) \geq f(p)$ *for* $k < i \leq d$.



Pole-Normal Lemma

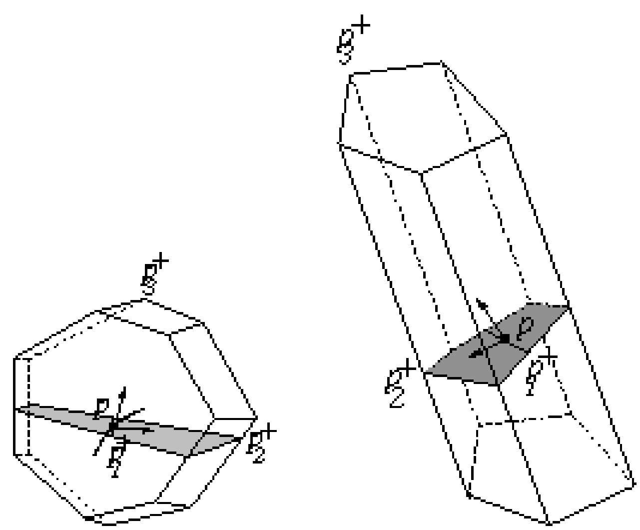
Lemma 3 $\exists \mathbf{n}(p) \in N(p)$ *for each pole vector* $\mathbf{v}(p,i)$, $k < i \leq d$ *so that* $\angle(\mathbf{v}(p,i), \mathbf{n}(p) \leq \alpha = 2\sin^{-1}\frac{\varepsilon}{1-\varepsilon}$.



Small-Height Lemma

Lemma 4 *Let $x \in bd(V(p, i))$ where $i \leq k$. We must have*

$$\frac{\delta}{2} f(p) \leq ||x - p|| \leq \frac{\varepsilon}{1 - \varepsilon} \sec\left(\frac{\alpha}{2}(1 + 4\sqrt{d - k})\right) f(p)$$



Height Theorem

Theorem 1 *Let $p \in P$ be any point derived from a manifold of dimension k embedded in $\mathbb{R}^d$ where P is an (ε, δ)-sample. Then following conditions hold:*

1. $H(p, i) \geq f(p)$ *for* $k < i \leq d$

2. $\frac{\delta}{2} f(p) \leq H(p, i) \leq \frac{\varepsilon}{1-\varepsilon} \sec(\frac{\alpha}{2}(1 + 4\sqrt{d-k})) f(p)$ *for* $1 \leq i \leq k$.

Ratio gap for ρ

$$
\begin{aligned}
\frac{H(p,1)}{H(p,i)} &\geq \frac{\delta(1-\varepsilon)}{2\varepsilon \sec\left(\frac{\alpha}{2}(1+4\sqrt{d-k})\right)} \\
&> \frac{(1-\varepsilon)}{4\sec\left(\frac{\alpha}{2}(1+4\sqrt{d-k})\right)} \\
&= \Omega(1) \text{ for } 1 \leq i \leq k,
\end{aligned}
$$

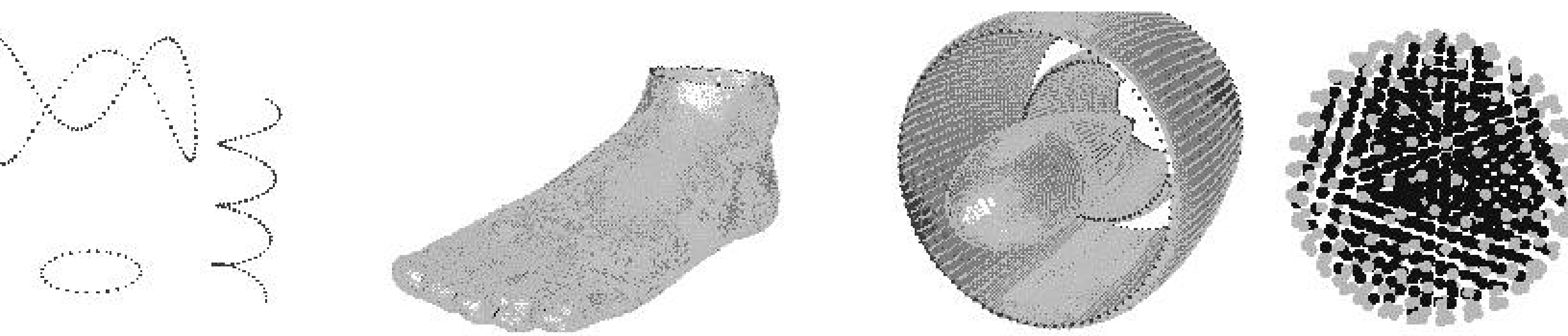
and

$$
\begin{aligned}
\frac{H(p,1)}{H(p,i)} &\leq \left(\frac{\varepsilon}{1-\varepsilon}\right) \sec\left(\frac{\alpha}{2}(1+4\sqrt{d-k})\right) \\
&= O(\varepsilon) \text{ for } k < i \leq d.
\end{aligned}
$$

Algorithm Dimension

```
DIMENSION (P, ρ, d)
Compute V(P)
for all p ∈ P
    compute H(p, 1), . . . , H(p, d)
    i := d
    while H(p, 1)/H(p, i) < ρ
        i := i − 1
    endwhile
    dim(p) := i
endfor
```

Experiments



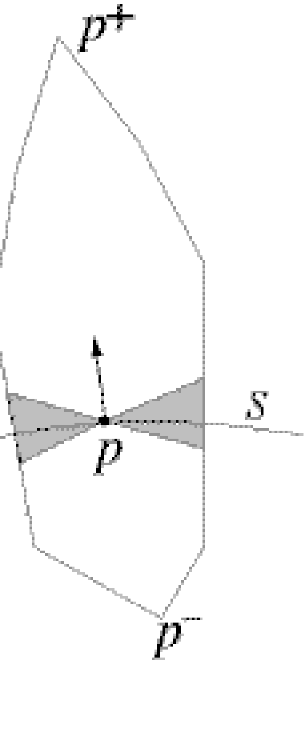
- CGAL library
- $\rho = 0.3$

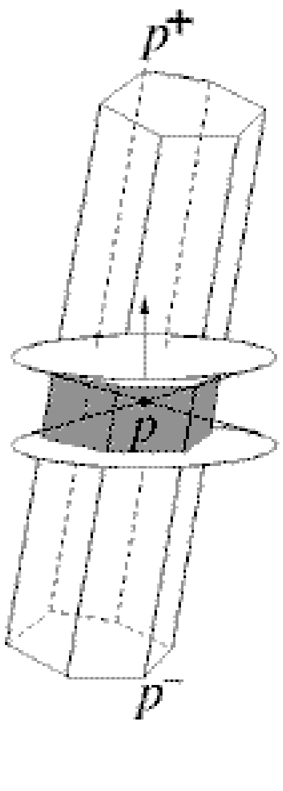
object	number of points	DIMENSION time (sec.)	COCONESHAPE time (sec.)
CURVES	259	2	2
BALL	805	4	1
CACTUS	3280	46	47
ENGINE	11360	237	138
TORUS	19090	466	42
FOOT	20021	110	106
SCENE	29285	227	189

Shape Reconstruction

- Compute a simplicial complex K with dist(|K|,M)=O(ε) times local feature size
- In R^2 and R^3, |K| and M are homeomorphic

Cocone C(p)





- C(p): x in V(p) makin angle < π/8 with V(p,k
- Compute all dual simplices to (d-k)-dimensional Voronoi edges intersecting C(p)

CoconeShape

COCONESHAPE (P, ρ)

Compute $V(P)$
DIMENSION(P)
$T := \emptyset$
for all $p \in P$
    Let $k := dim(p)$
    Compute $V(p, k)$ and $C(p)$
    Compute F, the $d - k$ dimensional Voronoi faces intersected by $C(p)$
    $T := T \cup dual(F)$
endfor
Output T

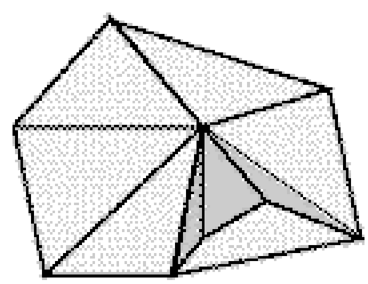
Shape Distance Theorem

eorem 2 *Each* $x \in |K|$ *is within* $O(\varepsilon)f(p)$ *from a* $p \in$

Follows from normal and small-height Lemma

Manifold Extraction(?)

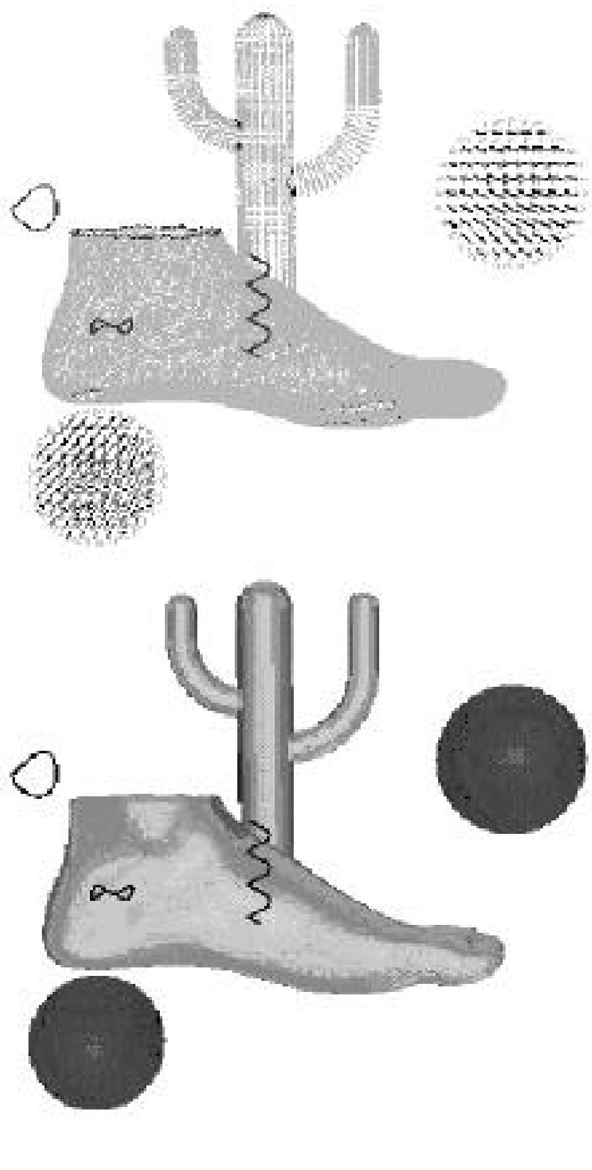
- How to extract a k-manifold out of K?
- Manifold extraction in R^3
  - Pruning
  - Walking



Homeomorphism(?)

?

Experimental Results



Result in R^4

- $w=x^2 + y^2 + z^2$
- 11×11×11 grid (1331 points)
- 3d points 615, 2d points 582, 1d points 134
- Ideally 637,578,116

Conclusions

- If k is known, CoconeShape works
- Boundaries can be handled?
- Manifold extraction?
- Immersion instead of embedding?
- Avoid Voronoi computation?