ORDINARY AND DIRECTED COMBINATORIAL HOMOTOPY FOR IMAGE ANALYSIS AND CONCURRENCY

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Abstract. We deal with intrinsic homotopy and homology theories for **simplicial complexes** and ***directed* simplicial complexes**.

Some applications are aimed at **image analysis in metric spaces** and have connections with digital topology and mathematical morphology; the directed version is applied to **directed images** and mathematical models of **concurrency**.

Previous works referred here

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[G2] M. Grandis, *Combinatorial homology and image analysis*, Dip. Mat. Univ. Genova, Preprint **394** (1999). http://www.dima.unige.it/~grandis/

[G3] M. Grandis, *Higher fundamental functors for simplicial sets*, Cahiers Top. Géom. Diff. Catég., to appear. http://arXiv.org/abs/math.AT/0009004

On simplicial complexes

[HW] P.J. Hilton - S. Wylie, *Homology theory*, Cambridge Univ. Press, Cambridge 1962.

[Sp] E.H. Spanier, *Algebraic Topology*, Mc Graw-Hill, New York 1966.

On digital topology

[KKM1] T.Y. Kong - R. Kopperman - P.R. Meyer, *A topological approach to digital topology*, Amer. Math. Monthly **98** (1991), 901-917.

[KKM2] T.Y. Kong - R. Kopperman - P.R. Meyer Eds., *Special issue on digital topology*, Topol. Appl. **46** (1992), no. 3, 173-303.

On mathematical morphology (e.g., dilation operators)

[FM] P. Frosini - M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bull. Belg. Math. Soc. Simon Stevin **6** (1999), 455-464.

[He] H.J.A.M. Heijmans, *Mathematical morphology: a modern approach in image processing based on algebra and geometry*, SIAM Rev. 37 (1995), 1-36.

On directed Algebraic Topology and concurrent processes

[FGR] L. Fajstrup - E. Goubault - M. Raussen, *Algebraic topology and concurrency*, Preprint 1999.

[Ga] P. Gaucher, *Homotopy invariants of higher dimensional categories and concurrency in computer science*, Math. Struct. in Comp. Science **10** (2000), 481-524.

[GG] P. Gaucher - E. Goubault, *Topological Deformation of Higher Dimensional Automata*, Preprint. http://www-irma.u-strasbg.fr/~gaucher/diCW.ps

[Pr] V. Pratt, *Higher dimensional automata revisited*, Math. Struct. in Comp. Science **10** (2000), 525-548.

1. Topological spaces, homotopy and homology // Images.

Topological Models for images (are not satisfactory)

- **topological space** X

Tools from classical Algebraic Topology

- $\pi_1(X, x_0)$: the **fundamental group** of X, at $x_0 \in X$
- $\Pi_1(X)$: the **fundamental groupoid** of X
- $H_1(X)$: the (singular) **homology group** of X, of degree 1

Applications to Images

"**continuous image**" $X \subset \mathbf{R}^2$

"**discrete image**" $X' = X \cap (\rho\mathbf{Z})^2$
(scanning at resolution $\rho = {}^1/_2$)

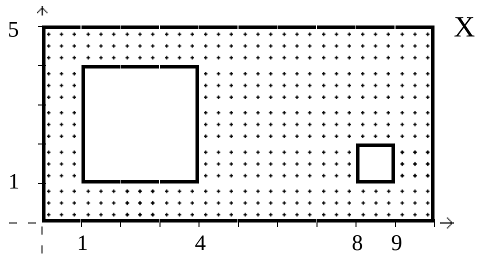


fig. (a)

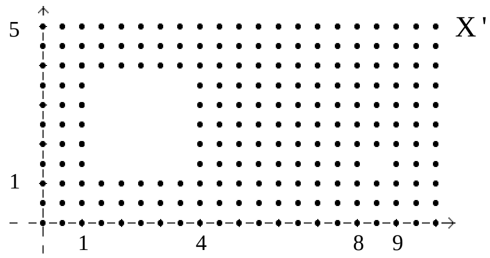


fig. (b)

Calculations

- $\pi_1(X, x_0) \;\cong\; \mathbf{Z} * \mathbf{Z}$
- $H_1(X) \;\cong\; \mathbf{Z} \oplus \mathbf{Z}$

– $\pi_1(X', x_0) = 0$

– $H_1(X') = 0$

Comments

- in fig. (a), homotopy and homology detect two "holes" (islands, basins,...)
- **but:** they ignore metric aspects
- **but:** in fig. (b), they give trivial information.

2. Metric spaces and simplicial complexes, combinatorial homotopy and homology

Metric Models for images (are better suited)

- **metric space** $X \subset \mathbf{R}^2$ (representing an *image*)
  $\mathbf{R}^2$: $d(x, y) = |x_1 - y_1| \vee |x_2 - y_2|$ (*l_∞-metric*, or *product metric*)

Tools from Combinatorial Algebraic Topology (references below)

homotopy and homology at resolution ε $(0 \le \varepsilon \le \infty)$

$t_\varepsilon X$: simplicial complex $\xi \subset X$ is *linked* iff: finite & $\mathrm{diam}(\xi) \le \varepsilon$

a **path** in $t_\varepsilon X$ "is" a finite sequence $(a_1,\dots a_p)$ with: $d(a_i, a_{i+1}) \le \varepsilon$

the **path functor** PX of simplicial complexes;

deformation of paths in P^2X; **combinatorial homotopies** $X \to PY$

homotopical equivalence: $t_1\mathbf{R}$ is contractible, by a *telescopic homotopy*

- $\pi_1^\varepsilon(X, x_0) = \pi_1(t_\varepsilon X, x_0)$: the fundamental group of X at x_0, *at resolution* ε
- $\Pi_1^\varepsilon(X) = \Pi_1(t_\varepsilon X)$: the fundamental groupoid of X, *at resolution* ε
- $H_1^\varepsilon(X) = H_1(t_\varepsilon X)$: the (singular) 1-homology group of X, *at resolution* ε.

References for these tools (classical or new)

- **Edge-path groupoid** $\Pi_1(K)$ of a simplicial complex: Spanier [SP], 3.6
- **Homology groups** $H_n(K)$ of a simplicial complex: [Sp], ch. 4; [HW], ch. 2
- **Higher homotopy groups** $\pi_n(K)$ of a (pointed) simplicial complex: [G1]
- intrinsic calculation of $\Pi_1(K)$, $\pi_1(K)$,
  by a "van Kampen" theorem and study of homotopies: [G1]
- **Higher homotopy groupoids** $\Pi_n(K)$ of a simplicial set: [G3]

  Π_n : **!Smp** $\rightleftarrows$ n-**Gpd** : M_n $\qquad$ $\Pi_n \dashv M_n$

  Π_n preserves all colimits (**strong** van Kampen property).

3. Applications to Images

$\mathbf{R}^2$: $d(x, y) = |x_1 - y_1| \vee |x_2 - y_2|$ (l_∞-*metric*, or *product metric*)

"**continuous image**" $X \subset \mathbf{R}^2$

"**discrete image**" $X' = X \cap (\rho\mathbf{Z})^2$
(scanning at resolution $\rho = {}^1/_2$)

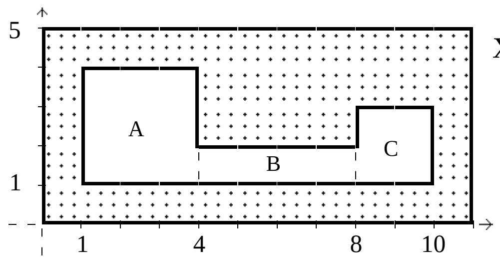


fig. (a)

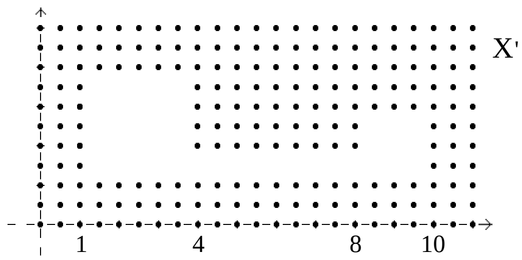


fig. (b)

– π_1^ε: **Z** $(0 < \varepsilon < 1;\ 2 \leq \varepsilon < 3)$

Z∗**Z** $(1 \leq \varepsilon < 2)$

{∗} $(3 \leq \varepsilon \leq \infty)$

same results, provided $\varepsilon \geq \rho = {}^1/_2$
(BUT: {∗} for $\varepsilon < \rho$)

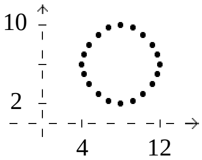
Comments

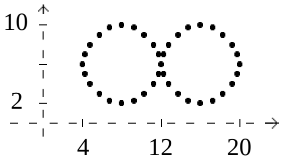
- at resolution $0 < \varepsilon < 1$: one single basin (or island...) (provided $\varepsilon \geq \rho$, for (b))
- $1 \leq \varepsilon < 2$: two basins A, C connected by a bridgeable channel B
  (or two islands connected by an isthmus, etc.) (B can be jumped over by paths)
- $2 \leq \varepsilon < 3$: one basin A with a negligible appendix
- $\varepsilon \geq 3$: no relevant basin at all.
- The choice of the resolution(s) of interest may be dictated by applications (e.g.: thresholds) but the finest description has been obtained at an intermediate resolution $(1 \leq \varepsilon < 2)$.

- The **whole analysis**, varying ε, is of interest. **Critical values**: 0, 1, 2, 3.

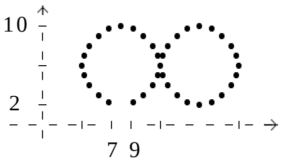
4. Applications to Images, continued

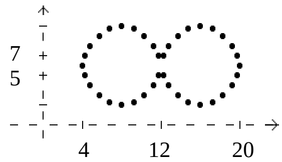
$\mathbf{R}^2$: $d(x, y) = |x_1 - y_1| \vee |x_2 - y_2|$ (l_∞-*metric*, or *product metric*)





– π_1^ε: **Z** $(1 \le \varepsilon < 8)$, $\{*\}$ $(\varepsilon \ge 8)$, **Z∗Z** $(1 \le \varepsilon < 8)$, $\{*\}$ $(\varepsilon \ge 8)$,





– π_1^ε: **Z** $(1 \le \varepsilon < 2)$, **Z∗Z** $(2 \le \varepsilon < 8)$, $\{*\}$ $(\varepsilon \ge 8)$.

Comments

- **a small resolution is sensitive to "errors"**

– the "best" description is obtained at an intermediate resolution $(2 \le \varepsilon < 8)$

5. Directed metric spaces, directed homotopy

Metric Models for directed images

- a **directed metric space** X (= a Lawvere generalised metric space [La])

  d-metric $\delta: X \times X \to [0, \infty]$ $\quad \delta(x, x) = 0, \quad \delta(x, y) + \delta(y, z) \geq \delta(x, z)$

  (X is a small category enriched over a suitable monoidal category with objects $t \in [0, \infty]$)

  F.W. Lawvere, *Metric spaces, generalized logic and closed categories*, Rend. Sem. Mat. Fis. Univ. Milano **43** (1974), 135-166. ~ quasi-pseudo-metric: J.C. Kelly, 1963.

- the category $\uparrow$**Mtr** of **directed metric spaces**, with functors

  $f_\varepsilon: \uparrow\mathbf{Mtr} \to \mathbf{Flw} \subset \uparrow\mathbf{Cs},$ $\quad x \prec_\varepsilon x' \Leftrightarrow \delta(x, x') \leq \varepsilon$

Tools

- **directed homotopy** at resolution ε $(0 \leq \varepsilon \leq \infty)$

  $f_\varepsilon X$: directed simplicial complex

  $\xi = (a_1,\dots a_p)$ is a *linked word* in $f_\varepsilon X$ iff: $i \leq j \Rightarrow a_i \prec_\varepsilon a_j$ $\quad (\delta(a_i, a_j) \leq \varepsilon)$

  a path in $t_\varepsilon X$ "is" a finite sequence $(a_1,\dots a_p)$ with: $\delta(a_i, a_{i+1}) \leq \varepsilon$

- $\uparrow\Pi_1^\varepsilon(X) = \uparrow\Pi_1(f_\varepsilon X)$: the fundamental *category* of X, *at resolution* ε
- $\uparrow\pi_1^\varepsilon(X, x_0) = \uparrow\Pi_1(f_\varepsilon X)(x_0, x_0)$: the fundamental *monoid* at x_0, *at resolution* ε

References for these tools (new)

- **Higher homotopy categories** $\uparrow\Pi_n(K)$ of a directed simplicial set: [G3]

  $\uparrow\Pi_n : \mathbf{!Smp} \rightleftarrows \mathbf{n\text{-}Gpd} : N_n$ $\qquad \uparrow\Pi_n \dashv N_n$

  $\uparrow\Pi_n$ preserves all colimits.

- ***Compare with:*** directed paths in a locally ordered space: [FGR, GG].

Elementary models: Flow metric spaces (preordered metric spaces)

- **metric space X with a flow relation** $x \prec x'$ (just reflexive)

  = a reflexive simple graph, with a metric on its set of vertices

- natural embedding

  $U: f\mathbf{Mtr} \to \uparrow\mathbf{Mtr},$ $\qquad (X, d, \prec) \mapsto (X, \delta)$

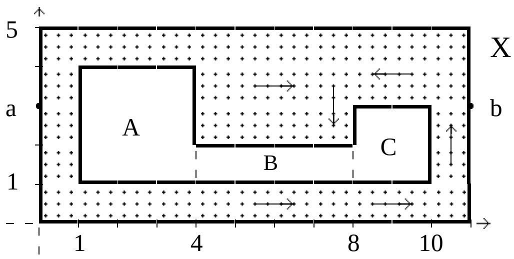
  $\delta(x, x') = \inf \{\Sigma_{i = 1,\dots n}\, d(x_{i-1}, x_i) \mid x = x_0 \prec x_1 \prec \dots \prec x_n = x'\}$

- **preordered case:** $\prec$ **transitive**

  $\delta(x, x') = d(x, x')$ if $x \leq x'$; $\quad = \infty$ otherwise

6. Applications to Directed Images

- **a 'flow image'** (modelled by a flow metric space)



$(x, y) \prec (x', y')$ iff:

$(x, x' \leq 5)$, *or* $(5 \leq x \leq x' \leq 7)$
or $(7 \leq x, x' \leq 11$ & $(x, y) \prec_C (x', y'))$

- $\prec_C$ is the counter-clockwise flow relation around C (cf. below)

Calculations

– $\uparrow\Pi_1^\varepsilon(X)(a, b)$:	1 arrow	$(0 < \varepsilon < 1)$
	$\|\mathbf{Z}\| \times \|\mathbf{N}\|$	$(1 \leq \varepsilon < 2)$
	$\|\mathbf{Z}\|$	$(2 \leq \varepsilon < 3)$
	$\{*\}$	$(3 \leq \varepsilon < \infty)$
– $\uparrow\Pi_1^\varepsilon(X)(b, a)$:	$\emptyset$	$(0 \leq \varepsilon < \infty)$

Comments

- at resolution $0 < \varepsilon < 1$: one stream
- for $1 \leq \varepsilon < 2$: two islands A, C, linked by a (broken) isthmus; the first in still water, the second with a vortex around
- for $2 \leq \varepsilon < 3$: one island A; then, for $\varepsilon \geq 3$, no relevant island;

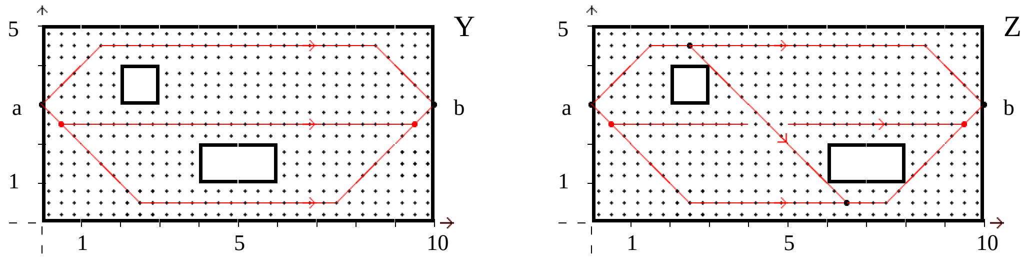
- Just fundamental **monoids** give insufficient information.

The counter-clockwise d-metric on $\uparrow\mathbf{S}^1$ (the directed circle)

- derives from the geodetic metric and a flow relation $x \prec_\varepsilon x'$ $(0 < \varepsilon \leq \pi)$
  - **different flow relations give the same d-metric**
  - **good maps are just locally monotone**
  - **flow relations are practically useful but theoretically unsatisfactory**

7. Applications to Directed Images, continued

- **An ordered image** (modelled by an ordered metric space)



$(x, y) \prec (x', y')$ iff $|y' - y| \leq x' - x$

Calculations

– $\uparrow\Pi_1^{\varepsilon}(Y)(a, b)$: 3 arrows $(0 < \varepsilon < 1)$, 1 arrow $(1 \leq \varepsilon < \infty)$

– $\uparrow\Pi_1^{\varepsilon}(Z)(a, b)$: 4 arrows $(0 < \varepsilon < 1)$, 1 arrow $(1 \leq \varepsilon < \infty)$

$\uparrow\Pi_1^{\varepsilon}(Y)(b, a) = \uparrow\Pi_1^{\varepsilon}(Z)(b, a) = \emptyset$

Comments (at resolution $0 < \varepsilon < 1$)

- two islands in a stream, at *comparable* levels (Y) or at *different* levels (Z)

Similar to the analysis in [FGR], fig. 14, for a model of execution paths of concurrent automata.

- **SPACE-TIME modelled by ordered metric spaces**

- **In the examples above:**

x: time, y: position in 1-dimensional space

$(x, y) \prec (x', y')$: one can go from (x, y) to (x', y'), with velocity ≤ 1

forbidden zones: obstacles in the line, having a limited duration in time

Classical model, with fixed frame ("material rest frame") and bounded velocity

(Also: relativistic model with fixed observer.)

8. Chu-spaces, flow sets, directed homotopy // concurrent processes

Models

– **Chu-spaces** as models for concurrency: [Pr]

(Directed homotopy for other models (cubical sets, l.o. spaces): [Ga, FGR, GG])

C = (A, r, X): **Chu-space** on $\boldsymbol{\Sigma = \{-1, 0, 1\} = \{-, 0, +\}}$
representing a concurrent process [PR]

A: *events* X: *states* r: A×X → Σ: the *matrix*

–, 0, +: past, present, future; or: unstarted, active, done

Tools

– the associated flow-set $f^*(C) = (X, \prec_1)$ (f^*: $\mathbf{Chu}_\Sigma \to \mathbf{Flw}^{op} \subset \uparrow\mathbf{Cs}^{op}$)

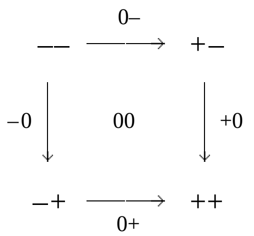
$x \prec_1 x' \iff 0 \le r(a, x') - r(a, x) \le 1 \ (\forall a \in A)$

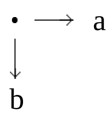
- $\uparrow\Pi_1(C) = \uparrow\Pi_1(X, \prec_1)$: the fundamental *category* of C

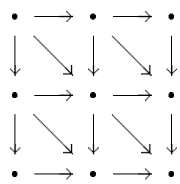
Applications to concurrent processes

I. The trivial case: the free Chu-space F(A) on a set of events A

$C = F(A) = (A, ev, \Sigma^A)$ e.g. A = {a, b}







$|\Sigma^A| = 3^2$ states: the 9 faces of the square the flow set $(X, \prec_1)$

Calculations

– $\uparrow\Pi_1(C)(x, y)$: 1 arrow if $x \le y$ $(r(u, x) \le r(u, x'),$ for $u = a, b)$

Ø otherwise.

Comments

- for a **free** Chu-space on a set of events the fundamental category is trivial:

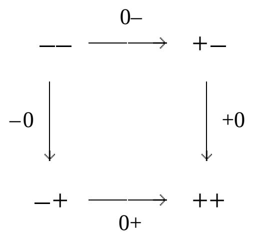
the order relation $\le$ generated by the flow $\prec_1$.

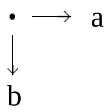
9. Applications to concurrent processes, continued

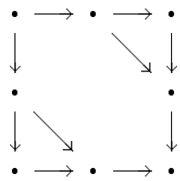
II. A non-trivial extensional case

$C = (A, r, X)$

extensional: $X \subset \Sigma^A$ (via $r^{\bullet}: X \to \Sigma^A$, $r^x = r(-, x)$)







Calculations

– $\uparrow\Pi_1(C)(--, ++)$: 2 arrows.

Comments

- C has **two** homotopy classes of execution paths, from –– to ++
  - the path through the pure state +– (*operate* a, *then* b)
  - the path through the pure state –+ (*operate* b, *then* a)
- they cannot be deformed one into the other,
  since the state "*operate* a *and* b" is missing (labelled 00 in previous case).

- An extensional (A, r, X) is *sculpted* from the free object $F(A)$, taking out unwanted states from Σ^A.
- $\uparrow\Pi_1(C)$ explores a flow subset of the $|A|$-dimensional cube $(\Sigma^A, \prec_1)$, no longer homotopically trivial (possibly).

10. Ordinary and symmetric simplicial sets and their cc-subcategories of simple presheaves

- **Presheaves and simple presheaves**

  $\hat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{op}}$ = category of presheaves on **C** (small category with terminal ⊤)

  a presheaf X is **simple** if every item $x \in X(c)$ is determined by its vertices $i^*(x)$ (i: ⊤ → c)

– Simple presheaves form a **cc-subcategory** of $\hat{\mathbf{C}}$

  i.e.: a full reflective subcategory whose reflector preserves finite products

  Freyd thm.: **this subcategory is cartesian closed, with the same exponentials**

- **Cc-embeddings** with their reflectors - - → (forgetful functors)

categories of presheaves	**simple presheaves** therein
Smp: simplicial sets	**↑Cs**: directed simplicial complexes
(presheaves on finite pos. ordinals)	**(sets with distinguished words)**
$\mathbf{Smp}_1$: graphs (oriented, reflexive)	**Flw**: flow sets (X, ≺) (= simple graphs)
!Smp: symmetric simplicial sets	**Cs**: simplicial complexes
(presheaves on finite pos. cardinals)	**(sets with distinguished subsets)**
$\mathbf{!Smp}_1$: involutive graphs	**Tol**: tolerance sets (X, !) (simp. inv. gph)

- **Definition: cartesian closed embedding**

  X is a **cc-category**; i: **A** → **X** is a **full reflective** subcategory with reflector p (p ⊣ i) which **preserves binary products**

- **Freyd theorem** (for cartesian closed embeddings)

– In these hypotheses, **A** is **cartesian closed**

  with $i(A^{pX}) = (iA)^X$. In particular, i preserves exponentials: $i(A^B) = (iA)^{iB}$.

  (P. Freyd, *Aspects of topoi*, Bull. Austral. Math. Soc. **7** (1972), 1-76.)